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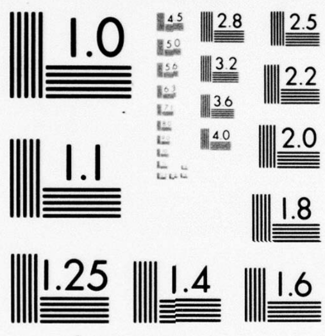

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We initiate the study of a family of models of military combat which include classical linear Lanchestrian models as a special case. These models introduce the additional concept of one or several columns providing supplies to one of the forces. It is assumed that the other force is defending an already supplied position or is prepared only for a short campaign. The models are designed so as to have solutions that are primarily qualitative in character.

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On Qualitative Lanchestrian Models of

Combat Incorporating Logistics.

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Abstract

We initiate the study of a family of models of military combat which include classical linear Lanchestrian models as a special case. These models introduce the additional concept of one or several columns providing supplies to one of the forces. It is assumed that the other force is defending an already supplied position or is prepared only for a short campaign. The models are designed so as to have solutions that are primarily qualitative in character. They have the form of a linear system of differential equations $\frac{du}{dt} = Au + F(t)$ where u is a vector with $m \geq 3$ components, A is an $m \times m$ constant matrix and F is a given time dependent vector with m components. The differential equations are to be solved subject to certain stopping rules. When we say that the solutions are qualitative in character, we mean that the mathematical form of the solution does not depend upon the magnitudes of the elements of the matrix A but only upon the signs of these elements. We introduce the concepts of upper and lower solutions which serve as bounds for all positive solutions of the problem. We obtain detailed results on the eigenvalues and eigenvectors of A and explicit representations of the upper and lower solutions.

1. Formulation of the Models

In this paper we introduce a class of Lanchestrian models of combat including logistic considerations and initiate the detailed analytic study of them. It turns out that the analysis of such models falls naturally into two parts, an analytic part in which we derive explicit solutions and examine their fundamental properties and a geometric part which provides valuable insights into the nature and behavior of solution trajectories. In order to keep this paper from becoming too lengthy we restrict ourselves herein to the analytic portion of the analysis reserving the geometric analysis for a subsequent paper.

Our models consist of two combat forces, an x - force either occupying an already supplied position or carrying enough supplies for a short (say a 30 day) campaign, and a y - force supplied by several columns at the levels $s_1,\ldots,s_p$. We assume that the equations governing the combat have the form

$$\left.\begin{aligned} \dot{x} &= -a_0x - b_1y - b_2s_1 - \cdots - b_{p+1}s_p + f(t) \\ \dot{y} &= -c_1x - a_1y + g(t) \\ \dot{s}_k &= -c_{k+1}x - a_{k+1}s_k + r_k(t)\ , \quad 1 \le k \le p\ . \end{aligned}\right\} \qquad (1)$$

In equations (1) the dots denote derivatives with respect to time. The coefficients a_0 and a_1 represent losses in the x and y forces not due to direct fire. They may be expanded to include such effects as losses due to desertion, sickness, and accidental injury as well as to interdiction by enemy fire power not itself subject to attrition. The coefficents b_1 and c_1 are the usual attrition-rate coefficents representing direct fire of the combat forces upon each other. The subsystem arising when $b_i = c_i = 0$, $2 \le k \le p + 1$, has been studied by

Bach, Dolansky, and Stubbs [1] .

Our analysis includes, of course, complete results for this subsystem.

The coefficients c_j , $2 \le j \le p+1$, represent the portion of the direct fire of the x - force used to interdict the supplies of the y - force. We assume that supplying the y - force at an adequate level enhances his ability to inflict casualties upon the x - force. The coefficients $b_2, \cdots, b_{p+1}$ capture this effect. The coefficients $a_2, \cdots, a_{p+1}$ represent losses to the supply columns not due to direct fire by the x - force. They may be expanded to represent the effects of equipment breakdown, adverse weather conditions or terrain, as well as losses to fire power not itself subject to attrition. Finally the nonhomogeneous terms $f(t)$, $g(t)$, $r_k(t)$, $1 \le k \le p$, represent reinforcements.

The system (1) is to be valid only for nonnegative functions x, y, s_k, $1 \le k \le p$, which are subject to certain stopping rules furnishing conditions under which one force or the other must disengage from combat. We consider two types of stopping rules.

Type 1 stopping rules: There exist constants $\beta > 0$, $\gamma \ge 0$, $\alpha_j > 0$, $1 \le j \le p$, such that the y - force must disengage if either

$$y \le \beta \tag{2}$$

or

$$y > \alpha_j s_j \ , \quad 1 \le j \le p \tag{3}$$

and the x - force must disengage if

$$x \le \gamma \ . \tag{4}$$

Type 2 stopping rules: There exists $\alpha > 0$ such that the y - force must disengage if either (2) holds or

$$y > \alpha(s_1^2 + \cdots + s_p^2)^{1/2} \quad , \tag{5}$$

and the x - force must disengage if (4) holds.

The condition (2) with $\beta > 0$ is included to avoid the paradoxical situation which would arise if $y = 0$ and at least one $s_j > 0$. Then we would have the x - force suffering losses inflicted by supplies alone.

The conditions (3) insure that the y - force is supplied at an adequate level. They may be regarded as the supply requirements when the several supply columns are furnishing different mixes of supplies. The condition (5) should be applied to the case where each supply column is furnishing approximately the same mix of items. Finally the condition (4) is an obvious stopping rule.

In addition to the stopping rules we have the initial conditions $x = x_0$, $y = y_0$, $s_j = s_{j0}$ at time $t = 0$ where all initial values are positive and

$$\left.\begin{array}{l} 0 < \beta < y_0 \\ \quad y_0 \le \alpha_j s_{j0} \ , \quad 1 \le j \le p \ , \\ 0 \le \gamma < x_0 \end{array}\right\} \tag{6}$$

if stopping rules of type 1 apply, and

$$\left.\begin{array}{l} 0 < \beta < y_0 \le \alpha(s_{10}^2 + \cdots + s_{p0}^2)^{1/2} \\ \quad 0 \le \gamma < x_0 \end{array}\right\} \tag{7}$$

if stopping rules of type 2 apply.

In most of our work we shall be concentrating on results that are primarily qualitative in nature. This is important because data regarding the magnitudes of the coefficients in the system (1) is either nonexistent or varies over such a large range of values that one must conclude either that it is unreliable or should be interpreted as stochastic. Thus we shall be interested in results which depend only upon the fact that the coefficents are either positive or zero, but which do not depend upon their magnitudes. Unfortunately, not all questions one might wish to ask about solutions are independent of the magnitudes of the coefficients. Nevertheless a remarkable number of properties of solutions are qualitative in character.

From the system (1) we may derive a second closely related system in the following way. Since we are concerned only with positive solutions of (1), let us introduce the numbers

$$\overline{a} = \max_{1\le k\le p+1} a_j ,$$

and

$$\underline{a} = \min_{1\le k\le p+1} a_j .$$

Then

$$\begin{aligned}
\overset{\circ}{x} &\le -a_0 x - b_1 y_0 - b_2 s_1 - \cdots - b_{p+1} s_p + f(t) , \\
\overset{\circ}{y} &\le -c_1 x - \underline{a} y + g(t) , \\
\overset{\circ}{s}_k &\le -c_{k+1} x - \underline{a} s_k + r_k(t) , \quad 1 \le k \le p ,
\end{aligned}$$

and

$$\begin{aligned}
\overset{\circ}{x} &\ge -a_0 x - b_1 y_0 - b_2 s_1 - \cdots - b_{p+1} s_p + f(t) , \\
\overset{\circ}{y} &\ge -c_1 x - \overline{a} y + g(t) , \\
\overset{\circ}{s}_k &\ge -c_{k+1} x - \overline{a} s_k + r_k(t) , \quad 1 \le k \le p .
\end{aligned}$$

It follows that positive solutions of the system

$$\overset{\circ}{x} = -a_0 x - b_1 y_0 - b_2 s_1 - \cdots - b_{p+1} s_p + f(t)$$
$$\overset{\circ}{y} = -c_1 x - ay + g(t) \qquad (8)$$
$$\overset{\circ}{s}_k = -c_{k+1} x - as_k + r_k(t) \;, \quad 1 \le k \le p \;,$$

furnish upper bounds for positive solutions of the system (1) when $a = \underline{a}$ and lower bounds for positive solutions of the system (1) when $a = \overline{a}$. It is therefore of obvious importance to carefully study properties of positive solutions of the system (8) .

The system (1) (and, of course, also the system (8)) has constant coefficients. It follows, therefore, that the analytic forms of solutions are completely determined by the eigenvalues and eigenvectors of the coefficient matrix A . In section 2 we show how to determine the multiplicities and the signs of the eigenvalues of this matrix. Our results are complete and detailed. We also completely detail the structure and properties of the corresponding eigenvectors. Generally speaking all of these results are independent of the magnitudes of the elements of the matrix A . The behavior of the largest eigenvalue of A , which is always simple, is an exception because it can be positive, negative, or zero depending upon the relative magnitudes of the elements of A . Thus we can obtain both stable and unstable solutions of the system (1) in various cases. The system (8) also exhibits the same possibilities.

Section 3 is short and summarizes how solutions are explicitly constructed using the eigenvalues and eigenvectors of A . We do this both for the homogeneous and for the nonhomogeneous case. This section really consists of the application of well-known results to the present problem.

In section 4 we explicitly construct the solution of the system (8) . For this special case the eigenvalues can be calculated as functions of the elements of A. This enables us to construct the fundamental matrix of A and its inverse. Rather simple analytic expressions are thus derived both for the solution of the homogeneous system and for the nonhomogeneous system.

In this paper we do not attempt to examine in detail the role played by the stopping rules. In a subsequent paper we will show how the stopping rules can be interpreted geometrically as boundaries of surfaces which contain the solution trajectories.

2. The Matrix of the System

Let us set $z = (x,y,s_1,\cdots,s_p)^T$ and define the matrix

$$A = \begin{bmatrix} -a_0 & -b_1 & -b_2 & \cdots & -b_{p+1} \\ -c_1 & -a_1 & 0 & \cdots & 0 \\ -c_2 & 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ -c_{p+1} & 0 & 0 & & -a_{p+1} \end{bmatrix} . \tag{9}$$

Then the system (1) can be written in the form

$$\dot{z} = Az + F(t) \tag{1a}$$

where

$$F(t) = (f(t),g(t),r_1(t),\ldots,r_p(t)) .$$

Obviously the evolution of the solution of system (1) or (1a) is determined by the properties of the $(p+2) \times (p+2)$ matrix A and the initial conditions. In this section we investigate the properties of A.

The matrix A belongs to the class J_2 (see [2]) which is a qualitatively determined class of matrices and, since symmetrically placed elements have the same sign, the matrix can be symmetrized (see [3]). In fact, for the diagonal matrix

$$D_+ = \operatorname{diag}[1,(b_1/c_1)^{1/2},(b_2/c_2)^{1/2},\cdots,(b_{p+1}/c_{p+1})^{1/2}] ,$$

we have $S(A) = D_+^{-1}AD_+$ a matrix of the form

$$S(A) = \begin{bmatrix} -a_0 & -(b_1c_1)^{1/2} & -(b_2c_2)^{1/2} & \cdots & -(b_{p+1}c_{p+1})^{1/2} \\ -(b_1c_1)^{1/2} & -a_1 & 0 & \cdots & 0 \\ -(b_2c_2)^{1/2} & 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ -(b_{p+1}c_{p+1})^{1/2} & 0 & 0 & & -a_{p+1} \end{bmatrix} .$$

Thus the matrix A is similar to a symmetric matrix, hence it has real eigenvalues and eigenvectors and is similar to a diagonal matrix.

We shall investigate more closely the eigenvalues of A. To this end note that

$$\det(\lambda I - A) = \prod_{i=0}^{p+1} (\lambda + a_i) - \sum_{j=1}^{p+1} b_jc_j \prod_{\substack{i=1 \\ i\neq j}}^{p+1} (\lambda + a_i) . \qquad (10)$$

We now parallel the development presented in the paper [4] (see also [5]).

Suppose there are k distinct values among the numbers a_i, $1 \le i \le p + 1$, which we take to be $\bar{a}_1,\cdots,\bar{a}_k$ in increasing order and suppose further that these occur with multiplicities $m_1,\cdots,m_k$ respectively where obviously we must have $m_1 + \cdots + m_k = p + 1$. (Note that $k = p + 1$ is a possible value in which case there are no repetitions among the numbers a_i, $1 \le i \le p + 1$.) From the righthand side of (10) we can factor out $\prod_{i=1}^{k}(\lambda + \bar{a}_i)^{m_i - 1}$, hence $-\bar{a}_i$ is an eigenvalue of A of multiplicity $m_i - 1$. Now divide $\det(\lambda I - A)$ by $\prod_{i=1}^{k} (\lambda + \bar{a}_i)^{m_i}$ to obtain

$$\frac{\det(\lambda I - A)}{\prod_{i=1}^{k} (\lambda + \bar{a}_i)^{m_i}} = \lambda + a_0 - \sum_{j=1}^{k} \frac{d_j}{\lambda + \bar{a}_j} = d(\lambda) + a_0 \quad ,$$

where

$$d(\lambda) = \lambda - \sum_{j=1}^{k} \frac{d_j}{\lambda + \bar{a}_j}$$

and d_j is the sum of the m_j values of $b_i c_i$ associated with $\bar{a}_j$. Clearly, the eigenvalues of A not found among the $\bar{a}_j$, $1 \le j \le k$, are going to satisfy

$$d(\lambda) = -a_0 \quad . \tag{11}$$

Let us plot $d(\lambda)$ against λ. Clearly, $d(\lambda)$ is asymptotic to each of the vertical lines $\lambda = \bar{a}_j$, $1 \le j \le k$. Moreover, we have

$$d'(\lambda) = 1 + \sum_{j=1}^{k} \frac{d_j}{(\lambda + \bar{a}_j)^2} > 0 \quad ,$$

hence d is an increasing function of λ wherever it is defined. Also $d(0) < 0$ if all $\bar{a}_i > 0$ and $d(0)$ is not defined if $\bar{a}_1 = 0$, $d \to \infty$ as $\lambda \to \infty$ and $d \to -\infty$ as $\lambda \to -\infty$. It follows that the graph is as shown below.

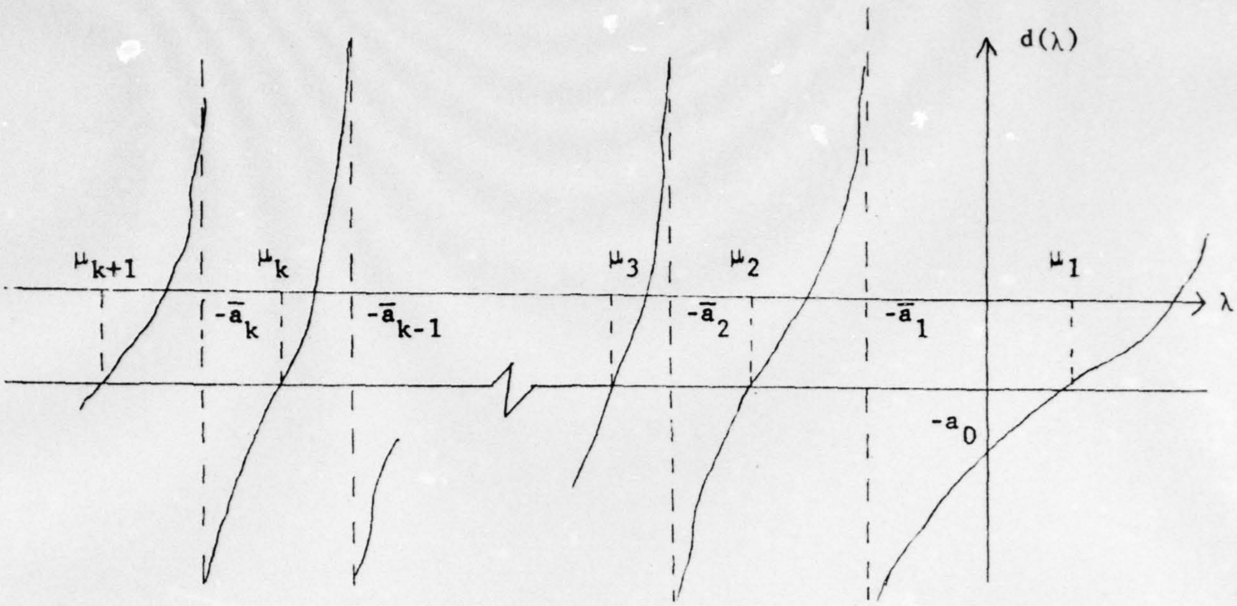

Figure 1

(In the case where $\bar{a}_1 = 0$ the vertical axis is a vertical asymptote.) We see that the $k + 1$ roots of (11), denoted by $\mu_1, \ldots, \mu_{k+1}$ satisfy the inequalities

$$\mu_1 > -\bar{a}_1 > \mu_2 > -\bar{a}_2 > \cdots > -\bar{a}_{k-1} > \mu_k > -\bar{a}_k > \mu_{k+1} . \qquad (12)$$

It follows that the $p + 2$ eigenvalues of A are divided into two groups as follows:

I. There are $p + 1 - k$ eigenvalues consisting of $m_i - 1$ values equal to $-\bar{a}_i$, $1 \le i \le k$.

II. There are $k + 1$ eigenvalues μ_j, $1 \le j \le k + 1$, which are simple and satisfy (12).

The first group can be empty (the case $k = p + 1$) but there are always at least two elements in the second group. In the case $k = 1$ where $a_1 = \ldots = a_{p+1} = \bar{a}_1$, the sequence of eigenvalues has the form

$$\mu_2 < -\bar{a}_1 < \mu_1$$

with $\bar{a}_1$ occurring p times.

It is important to observe that the largest eigenvalue μ_1 can be positive, negative, or zero (the case $\mu_1 > 0$ is illustrated in figure 1). Obviously, if $a_0 = 0$ or $\bar{a}_1 = 0$ then $\mu_1 > 0$, but if $\bar{a}_1 < 0$ and $a_0 > 0$ all cases can occur. Moreover, we can formulate the conditions for $\lambda = 0$ to satisfy (11). In fact, $d(0) = -\sum_{j=1}^{k} \frac{d_j}{\bar{a}_j}$, so $\mu_1 = 0$ is an eigenvalue if

$$a_0 = \sum_{j=1}^{k} \frac{d_j}{\bar{a}_j} \tag{13a}$$

$\mu_1 < 0$ if

$$a_0 > \sum_{j=1}^{k} \frac{d_j}{\bar{a}_j} , \tag{13b}$$

and $\mu_1 > 0$ if

$$a_0 < \sum_{j=1}^{k} \frac{d_j}{\bar{a}_j} . \tag{13c}$$

In the case where $\mu_1 < 0$ the matrix A is a stable matrix, otherwise A is not stable. We shall see subsequently that the behavior of solutions of the system (1) are not qualitatively different in the stable case from their behavior in the nonstable case.

It is instructive to examine two specific examples of the class of matrices A.

<u>Example 1.</u> Consider the case $p = 1$ so that $(a_0 > 0)$

$$A = \begin{bmatrix} -a_0 & -b_1 & -b_2 \\ -c_1 & -a_1 & 0 \\ -c_2 & 0 & -a_2 \end{bmatrix} .$$

We have $\det(\lambda I - A) = (\lambda + a_0)(\lambda + a_1)(\lambda + a_2) - b_1c_1(\lambda + a_2) - b_2c_2(\lambda + a_1)$.

Thus two basic cases can arise. First, if $a_1 = a_2 = \bar{a}_1$, we have

$\det(\lambda I - A) = (\lambda + \bar{a}_1)^2[\lambda + a_0 - \frac{b_1c_1 + b_2c_2}{\lambda + \bar{a}_1}]$. It follows that $-\bar{a}_1$ is a simple eigenvalue of A whose three eigenvalues satisfy

$$\mu_2 < -\bar{a}_1 < \mu_1 .$$

For $\bar{a}_1 = 0$, $\mu_1 > 0$. Otherwise A is a stable matrix if

$$a_0 > \frac{b_1c_1 + b_2c_2}{\bar{a}_1}$$

and A is not stable if $a_0 \le \frac{b_1c_1 + b_2c_2}{\bar{a}_1}$. Next if $a_1 \neq a_2$ and $\min(a_1, a_2) < 0$, we have

$$-\mu_3 < -\max(a_1, a_2) < \mu_2 < -\min(a_1, a_2) < \mu_1 .$$

Now A is a stable matrix if

$$a_0 > \frac{b_1c_1}{a_1} + \frac{b_2c_2}{a_2}$$

and A is not stable if

$$a_0 \le \frac{b_1c_1}{a_1} + \frac{b_2c_2}{a_2} .$$

Note that multiple eigenvalues cannot occur when $p = 1$.

<u>Example 2.</u> Consider next the case $p = 2$ (again assuming $a_0 > 0$).

We have

$$A = \begin{bmatrix} -a_0 & -b_1 & -b_2 & -b_3 \\ -c_1 & -a_1 & 0 & 0 \\ -c_2 & 0 & -a_2 & 0 \\ -c_3 & 0 & 0 & -a_3 \end{bmatrix}$$

so that

$$\det(\lambda I - A) = (\lambda + a_0)(\lambda + a_1)(\lambda + a_2)(\lambda + a_3) - b_1c_1(\lambda + a_2)(\lambda + a_3)$$
$$- b_2c_2(\lambda + a_1)(\lambda + a_3) - b_3c_3(\lambda + a_1)(\lambda + a_2) .$$

There are now essentially three cases.

Case 1. $a_1 = a_2 = a_3 = \bar{a}_1 > 0$. Then

$$\det(\lambda I - A) = (\lambda + \bar{a}_1)^3[\lambda + a_0 - \frac{b_1c_1 + b_2c_2 + b_3c_3}{\lambda + \bar{a}_1}]$$

and the eigenvalues of A satisfy

$$\mu_2 < -\bar{a}_1 < \mu_1$$

with $-\bar{a}_1$ occurring twice as an eigenvalue. A is stable if

$$a_0 > \frac{b_1c_1 + b_2c_2 + b_3c_3}{\bar{a}_1} ,$$

and nonstable if

$$a_0 \le \frac{b_1c_1 + b_2c_2 + b_3c_3}{\bar{a}_1} .$$

Case 2. Two of the numbers a_1, a_2, a_3 are equal and the third different, say, $a_1 = a_3 = \bar{a}_2, a_2 = \bar{a}_1$. Now

$$\det(\lambda I - A) = (\lambda + \bar{a}_1)(\lambda + \bar{a}_2)^2[\lambda + a_0 - \frac{b_1c_1 + b_3c_3}{\lambda + \bar{a}_2} - \frac{b_2c_2}{\lambda + \bar{a}_1}]$$

and the eigenvalues of A satisfy

$$\mu_3 < -\bar{a}_3 < \mu_2 < -\bar{a}_1 < \mu_1$$

with $-\bar{a}_2$ a simple eigenvalue. This time A is stable if and only if

$$a_0 > \frac{b_1c_1 + b_3c_3}{\bar{a}_2} + \frac{b_2c_2}{\bar{a}_1} .$$

<u>Case 3.</u> The numbers a_1, a_2, a_3 are distinct with, say, $a_2 < a_1 < a_3$, hence

$$\det(\lambda I - A) = (\lambda + a_1)(\lambda + a_2)(\lambda + a_3)[\lambda + a_0 - \frac{b_1c_1}{\lambda + a_1} - \frac{b_2c_2}{\lambda + a_2} - \frac{b_3c_3}{\lambda + a_3}]$$

so the eigenvalues of A satisfy

$$\mu_4 < a_3 < \mu_3 < a_1 < \mu_2 < a_2 < \mu_1 .$$

Now A is stable if and only if $a_0 > \frac{b_1c_1}{a_1} + \frac{b_2c_2}{a_2} + \frac{b_3c_3}{a_3}$.

Let us turn next to the eigenvectors of A. We must consider the equation $(\lambda I - A)\tilde{u} = 0$ where $\tilde{u} = (u_1, \cdots, u_n)^T$ and λ is an eigenvector of A. This system is

$$(\lambda + a_0)u_1 + \sum_{j=1}^{p+1} b_j u_{j+1} = 0 , \tag{14}$$

$$c_j u_1 + (\lambda + a_j)u_{j+1} = 0 . \tag{15}$$

Suppose first that $\lambda = \mu_j$, $1 \le j \le k+1$, then $\lambda + a_j \ne 0$ for $1 \le j \le p+1$, and from (15) we obtain

$$u_i = -\frac{c_{i-1}}{\mu_j + a_{i-1}} u_1 , \quad 2 \le i \le p+2 .$$

Choosing $u_1 = 1$, we obtain the eigenvectors

$$\tilde{u}_j = (1, -\frac{c_1}{\mu_j + a_1}, -\frac{c_2}{\mu_j + a_2}, \cdots, -\frac{c_{p+1}}{\mu_j + a_{p+1}})^T , \tag{16}$$

$1 \le j \le k+1$. Now $c_i > 0$, $1 \le i \le p+1$, hence the signs of the elements of the eigenvectors depend only upon the denominators of the components. For $j = 1$ we have $\mu_j + a_i > 0$ for $1 \le i \le p+1$, hence every component is negative except for the first component. Thus $\tilde{u}_1$ has the sign pattern $(+,-,-,\cdots,-)$. On the other hand for

$j = k + 1$ we have $\mu_{k+1} + a_i < 0$ for $1 \le i \le p + 1$, so that $\tilde{u}_{k+1}$ has the sign pattern $(+,+,+,\cdots,+)$. Finally, consider $\tilde{u}_j$, $1 < j < k + 1$. We have $\mu_j + a_i < 0$ for $m_1 + \cdots + m_{j-1}$ values of a_i and $\mu_j + a_i > 0$ for $m_j + \cdots + m_k$ values of a_i, $1 \le i \le p + 1$. Thus each component of $\tilde{u}_j$ for which a_i has one of the values $\bar{a}_1,\cdots,\bar{a}_{j-1}$ will be positive and each component for which a_i has one of the values $\bar{a}_j,\cdots,\bar{a}_k$ will be negative.

Next suppose $\lambda = -\bar{a}_i$ for some $1 \le i \le k$. Then m_i equations of the system (15) reduce to the form $c_i u_1 = 0$, hence $u_1 = 0$. The remaining $p + 1 - m_i$ equations from this subsystem thus reduce to $(-\bar{a}_i + a_j)u_{j+1} = 0$. Since $(-\bar{a}_i + a_j) \neq 0$, it follows that the corresponding $u_j = 0$. Finally the m_i components which need not be zero must satisfy (14), which becomes

$$\sum_{j=2}^{p+1} b_j u_{j+1} = 0 \tag{14a}$$

because $u_1 = 0$. There are m_i nonzero terms on the lefthand side of (14a), and we can clearly obtain $m_i - 1$ linearly independent solutions, hence $m_i - 1$ linearly independent eigenvectors. In fact, it is easy to see each of these eigenvectors can be chosen so as to have exactly two nonzero components, one positive and one negative.

In order to see exactly how the eigenvectors appear in the various possible cases let us find them in the special case $p = 2$.

<u>Example 2 (continuation).</u> Consider first the eigenvectors for case 1, $a_1 = a_2 = a_3 = \bar{a}_1 > 0$. Corresponding to μ_1 and μ_2 we have the eigenvectors

$$u_i = \left(1, -\frac{c_1}{\mu_i + \bar{a}_1}, -\frac{c_2}{\mu_i + \bar{a}}, -\frac{c_3}{\mu_i + \bar{a}_1}\right), \quad i = 1,2,$$

while corresponding to the double eigenvalue $-\bar{a}_1$ we have the eigenvectors

$$u_3 = (0,b_2,-b_1,0) ,$$
$$u_4 = (0,b_3,0,-b_1) .$$

Turning next to the case 2, we have the eigenvectors u_1, u_2, u_3 given by

$$u_i = (1, -\frac{c_1}{\mu_i+\bar{a}_2}, -\frac{c_2}{\mu_i+\bar{a}_1}, -\frac{c_3}{\mu_i+\bar{a}_2}) , \quad i = 1,2,3$$

and the eigenvector

$$u_4 = (0,b_3,0,-b_1) .$$

Finally, in case 3 all eigenvectors $u_1,\cdots,u_4$ have the form

$$u_i = (1, -\frac{c_1}{\mu_i+a_1}, -\frac{c_2}{\mu_i+a_2}, -\frac{c_3}{\mu_i+a_3}) , \quad 1 \le i \le 4 .$$

We wish to point out that one of the consequences of our analysis to this point is that multiple eigenvalues can occur only in the event that there are two or more supply columns. This means that the matrix A must be at least of order 4.

It is of considerable interest to understand how the eigenvalues of A vary when the elements of A vary. To this end consider first the element a_0. It is clear from figure 1 that increasing a_0 will decrease each of the eigenvalues μ_j, $1 \le j \le k + 1$. Thus we can assert that

$$\frac{d\mu_j}{da_0} < 0 , \quad 1 \le j \le k + 1 . \tag{17}$$

On the other hand, it is also clear that the eigenvalues $-\bar{a}_i$, $1 \le i \le k$, do not depend upon a_0, i.e.,

$$\frac{d(-\bar{a}_i)}{da_0} = 0 \ , \quad 1 \le i \le k \quad . \tag{18}$$

Thus we have complete information regarding the dependence of the eigenvalues upon the element a_0 .

3. Construction of Solutions

Now it is well known (see [6], [7]) that the solution of the homogeneous system (1a) can be written in the form

$$z(t) = e^{tA} z(0) \tag{19}$$

where $z(0) = (x_0, y_0, s_{10}, \cdots, s_{p0})^T$ is the vector of initial values at time $t = 0$. The matrix $\exp(tA)$ is called a fundamental matrix of the system. Let us determine the form of this matrix.

To this end let U be the matrix whose column vectors are the eigenvectors of A . Then

$$AU = UJ$$

where J is the Jordan matrix of A . Since A is similar to a symmetric matrix, J is a diagonal matrix with diagonal elements the eigenvalues $\mu_1, \cdots, \mu_j$, and $-\bar{a}_1, \cdots, -\bar{a}_k$, the latter occurring with appropriate multiplicities. Thus, denoting the eigenvalues of A by $\lambda_1, \cdots, \lambda_n$, we have

$$J = \text{diag}[\lambda_1, \cdots, \lambda_n] \ .$$

Now we have

$$e^{tA} = e^{tUJU^{-1}} = Ue^{tJ}U^{-1} \ ,$$

where

$$e^{tJ} = \mathrm{diag}[e^{\lambda_1 t}, \cdots, e^{\lambda_n t}] \ .$$

More explicitly, then

$$e^{tA} = U\ \mathrm{diag}[e^{\mu_1 t}, \cdots, e^{\mu_{k+1} t}, e^{-\bar{a}_1 t}, \cdots, e^{-\bar{a}_k t}]U^{-1} \ .$$

It follows, therefore, that (19) appears in the form

$$z(t) = U\ \mathrm{diag}[e^{\mu_1 t}, \cdots, e^{-\bar{a}_k t}]U^{-1}z(0) \ . \tag{19'}$$

Let us examine this formula more closely. First consider the matrix $U\ \mathrm{diag}[e^{\mu_1 t}, \cdots, e^{-\bar{a}_k t}]$. Denote this matrix by $U(t)$. Multiplication of any matrix on the right by a diagonal matrix multiplies the columns of the matrix by the corresponding diagonal elements. Thus the first $k + 1$ column vectors of $U(t)$ are just the vectors

$$\tilde{u}_j(t) = \left(e^{\mu_j t}, -\frac{c_1 e^{\mu_j t}}{\mu_j + a_1}, \cdots, -\frac{c_{p+1} e^{\mu_j t}}{\mu_j + a_{p+1}}\right)^T , \tag{20}$$

$1 \le j \le k + 1$. The remaining columns of U have the form

$$\tilde{u}_k(t) = (u_{k1} e^{-\bar{a}_i t}, \cdots, a_{kp+2} e^{-\bar{a}_i t}) \tag{21}$$

where exactly two of the numbers u_{kj} , $1 \le j \le p + 2$ are different from zero and each $\bar{a}_i$ is one of the numbers $\bar{a}_1, \cdots, \bar{a}_k$.

Next we observe that $U^{-1}z(0)$ is a vector. Let us denote it by w . Then the formula (19) can be rewritten as

$$z(t) = U(t)w \ . \tag{22}$$

Let us turn next to the solution of the nonhomogeneous system

$$\dot{z} = Az + F(t) \tag{23}$$

where $f(t) = (f(t), g(t), r_1(t), \cdots, r_p(t))$. As mentioned above we may think of the vector F as representing reinforcements. In the case of the x and y forces this might be additional personnel, while for the supply columns it might consist of vehicles or other equipment replacements. The general formula for solving the system (23) is

$$z(t) = e^{tA}z(0) + \int_0^t e^{(t-s)A}F(s)ds \quad . \tag{24}$$

In view of the form which we have presented above for the fundamental matrix of the system, we can rewrite (24) as

$$z(t) = U(t)w + \int_0^t U(t-s)U^{-1}F(s)ds \; . \tag{25}$$

It is evident from the formulas (22) and (25) that we must investigate the form of the matrix U^{-1} . We shall not attempt to do this for the most general case.

4. Upper and Lower Solutions

As was pointed out in section 1 the system (8) is of particular importance because its solutions for appropriate values of a can be used to obtain both upper and lower bounds for all positive solutions of the system (1) . In this section we shall explicitly solve the system (8) .

Thus consider the matrix

$$A_0 = \begin{bmatrix} -a_0 & -b_1 & \dots & -b_{p+1} \\ -c_1 & -a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_{p+1} & 0 & \dots & -a \end{bmatrix} .$$

The characteristic polynomial of A_0 is

$$\det(\lambda I - A_0) = (\lambda + a)^p[\lambda^2 + (a_0 + a)\lambda + aa_0 - \delta^2] ,$$

where $\delta = \left(\sum_{j=1}^{p+1} b_j c_j\right)^{1/2}$. Therefore the eigenvalues of A_0 are

$$\lambda = -a , \quad p - \text{times} ,$$

$$\lambda = -\frac{a_0 + a}{2} + \frac{1}{2}[(a_0 - a)^2 + 4\delta^2]^{1/2} ,$$

and

$$\lambda = -\frac{a_0 + a}{2} - \frac{1}{2}[(a_0 - a)^2 + 4\delta^2]^{1/2} .$$

Let us set

$$\varepsilon = \frac{1}{2}\sqrt{(a_0 - a)^2 + 4\delta^2} ,$$

$$\sigma = -\frac{a_0 + a}{2}$$

so that the eigenvalues of A_0 become

$$\left.\begin{aligned} &\lambda = -a, \ p - \text{times} , \\ &\lambda = \sigma + \varepsilon \\ &\lambda = \sigma - \varepsilon . \end{aligned}\right\} \qquad (26)$$

Now the fundamental matrix corresponding to A_0 is

$$U = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ -\frac{c_1}{\sigma + \varepsilon + a} & -\frac{c_1}{\sigma - \varepsilon + a} & b_2 & \cdots & b_{p+1} \\ & & -b_1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\frac{c_{p+1}}{\sigma + \varepsilon + a} & -\frac{c_{p+1}}{\sigma - \varepsilon + a} & 0 & & -b_1 \end{bmatrix} \qquad (27)$$

It is easy to show that $\det U = (-1)^p 2b_1^{p-1}\varepsilon$ and one obtains

$$U^{-1} = \begin{bmatrix} -\frac{\delta^2}{2\varepsilon(\sigma-\varepsilon+a)} & -\frac{b_1}{2\varepsilon} & -\frac{b_2}{2\varepsilon} & -\frac{b_3}{2\varepsilon} & \cdots & -\frac{b_{p+1}}{2\varepsilon} \\ \frac{\delta^2}{2\varepsilon(\sigma+\varepsilon+a)} & \frac{b_1}{2\varepsilon} & \frac{b_2}{2\varepsilon} & \frac{b_3}{2\varepsilon} & \cdots & \frac{b_{p+1}}{2\varepsilon} \\ 0 & \frac{c_2}{\delta^2} & -\frac{\delta_2}{\delta^2 b_1} & \frac{c_2 b_3}{\delta^2 b_1} & \cdots & \frac{c_2 b_{p+1}}{\delta^2 b_1} \\ 0 & \frac{c_2}{\delta^2} & \frac{b_2 c_3}{\delta^2 b_1} & -\frac{\delta_3}{\delta^2 b_1} & \cdots & \frac{c_3 b_{p+1}}{\delta^2 b_1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{c_{p+1}}{\delta^2} & \frac{b_2 c_{p+1}}{\delta^2 b_1} & \frac{b_3 c_{p+1}}{\delta^2 b_1} & \cdots & -\frac{\delta_{p+1}}{\delta^2 b_1} \end{bmatrix}$$

where we have defined

$$\delta_j = \sum_{i=1,\, i\neq j}^{p+1} b_i c_i \ , \quad 2 \le j \le p+1 \ .$$

Next let us define

$$D_0 = b_1 y_0 + \sum_{j=2}^{p+1} b_j s_{j-1,0}$$

and compute the components of the vector w. After some calculations one finds

$$w_1 = \frac{1}{2\varepsilon}\left[-\frac{\delta^2}{\sigma-\varepsilon+a} x_0 - D_0\right] \ ,$$

$$w_2 = \frac{1}{2\varepsilon}\left[\frac{\delta^2}{\sigma+\varepsilon+a} x_0 + D_0\right] \ ,$$

$$w_k = b_1^{-1}[-s_{k-2,0} + c_{k-1}\delta^{-2} D_0] \ , \quad 3 \le k \le p+2 \ .$$

With the vector w at hand we can write out the solution to the homogeneous version of the system (8). It is

$$\left.\begin{aligned}
x(t) &= e^{\sigma t}\{x_0 \cosh \varepsilon t + \tfrac{1}{\varepsilon}[(\sigma + a)x_0 - D_0]\sinh \varepsilon t\} \ , \\
y(t) &= (y_0 - \frac{c_1 D_0}{\delta^2})e^{-at} - \frac{c_1}{\varepsilon}[x_0 + \frac{(\sigma + a)D_0}{\delta^2}]e^{\sigma t} \sinh \varepsilon t + \frac{c_1 D_0}{\delta^2} e^{\sigma t} \cosh \varepsilon t, \\
s_k(t) &= (s_{k,0} - \frac{c_{k+1} D_0}{\delta^2})e^{-at} - \frac{c_{k+1}}{\varepsilon}[x_0 + \frac{(\sigma + a)D_0}{\delta^2}]e^{\sigma t} \sinh \varepsilon t \\
&\qquad + \frac{c_{k+1} D_0}{\delta^2} e^{\sigma t} \cosh \varepsilon t \ , \quad 1 \le k \le p \ .
\end{aligned}\right\} \quad (28)$$

Now observe that the stable case occurs if and only if

$$a a_0 - \delta^2 > 0 \ . \qquad (29)$$

But we also note that in any event $a \ge 0$, $\sigma < 0$, and $\varepsilon > 0$. Consequently the analytic form of the solution (28) is the same whether or not A_0 *is stable.* This is attributable to the fact that only the sign of the eigenvalue $\lambda = \sigma + \varepsilon$ is affected by the change from the stable to the unstable case. However, we also wish to call special attention to the situation occurring when $a = 0$. Then the solution will always be unstable because (29) reduces to $-\delta^2 > 0$ which can never be satisfied.

Let us next write out the solution to the nonhomogeneous problem. To this end it is sufficient to write out the integral terms appearing in equation (25) because these terms added to the corresponding terms in (28) yield the entire solution.

Set $\tilde{w} = U^{-1}F(s)$. Then we have

$$\tilde{w}_1 = \frac{1}{2\varepsilon}[-\frac{\delta^2}{\sigma - \varepsilon + a}f(s) - D_0(s)] \ ,$$

$$\tilde{w}_2 = \frac{1}{2\varepsilon}[\frac{\delta^2}{\sigma + \varepsilon + a}f(s) + D_0(s)] \ ,$$

$$\tilde{w}_k = b_1^{-1}[-r_{k-2}(s) + c_{k-1}\delta^{-2}D_0(s)] \ ,$$

where

$$D_0(s) = b_1 g(s) + \sum_{j=2}^{p+1} b_j r_{j-1}(s) \ . \tag{30}$$

Now let $(\tilde{x}(t), \tilde{y}(t), \tilde{s}_1(t), \cdots, \tilde{s}_p(t))$ denote the solution of the nonhomogeneous system satisfying zero initial conditions. We then obtain

$$\left.\begin{aligned}
\tilde{x}(t) &= \int_0^t e^{\sigma(t-s)}\{f(s)\cosh \varepsilon(t-s) + [\frac{\sigma+a}{\varepsilon}f(s) - \frac{D_0(s)}{\varepsilon}]\sinh \varepsilon(t-s)\}ds \ , \\
\tilde{y}(t) &= \int_0^t (g(s) - \frac{c_1}{\delta^2} D_0(s))e^{-a(t-s)}ds \\
&\quad + \int_0^t [\frac{c_1}{\delta^2} D_0(s)e^{\sigma(t-s)}\cosh \varepsilon(t-s) - \frac{c_1}{\varepsilon}[f(s) + \frac{\sigma+a}{\delta^2}D_0(s)]e^{-\sigma(t-s)} \times \\
&\qquad \times \sinh \varepsilon(t-s)]ds \ , \\
\tilde{s}_k(t) &= \int_0^t (r_k(s) - \frac{c_{k+1}}{\delta^2}D_0(s))e^{-a(t-s)}ds + \int_0^t [\frac{c_{k+1}}{\delta^2}D_0(s)e^{\sigma(t-s)}\cosh \varepsilon(t-s) \\
&\qquad - \frac{c_{k+1}}{\varepsilon}[f(s) + \frac{\sigma+a}{\delta^2}D_0(s)]e^{-\sigma(t-s)}\sinh \varepsilon(t-s)]\, ds, \\
&\qquad 1 \le k \le p \ .
\end{aligned}\right\} \tag{31}$$

Again because $a \geq 0$, $\sigma < 0$, and $\varepsilon > 0$ this portion of the solution has the same form in both the stable and the nonstable case.

The formulas (28) and (31) provide the basic tools for any additional analytic analysis of the system (8). On the other hand, as was pointed out in the introduction, the solutions we have obtained provide upper and lower bounds for all solutions of the system (1). These bounds will hold as long as the bounding solutions remain positive and do not violate the stopping rules. Since the bounding solutions have the same initial values as the solutions we wish to bound, either the upper bound or the lower bound may be expected to remain in force until the solution violates one c the stopping rules. In view of the fact that both bounding solutions have the same mathematical form we expect that they will furnish reasonably close estimates of the actual solution. This is, of course, only a conjecture. We expect to examine more closely the behavior of the solutions of our model in a subsequent paper.

In view of the fact that our aim in this paper was to initiate the study of models whose dependence upon the model parameters is primarily qualitative it is gratifying to observe that the upper and lower solutions both have the same structure independently of the magnitudes of the elements of A as long as all of the numbers a_i, $0 \le i \le p + 1$, b_i, c_i, $1 \le i \le p + 1$, are positive. This is about the strongest qualitative result we could hope to obtain. Moreover, the form of the solution in the case where $a_i \ge 0$, $0 \le i \le p + 1$ is also easily deduced from the above results.

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